归纳与递归

回顾

问题1：什么是(初等)数论?

- 研究整数的性质：整除、余数、同余算术

问题2：素数有哪些性质?

- 素数基本定理
- 最大公约数
- 同余方程
- 欧拉定理/费马小定理

本节提要

- 归纳：
  - 数学归纳法与强数学归纳法
  - 运用良序公理来证明
- 递归：
  - 递归定义
  - 结构归纳法与递归算法

数学归纳法

- 证明目标
  - $\forall n\ P(n)$　//n的论域为正整数集合
- 证明框架
  - 基础步骤：$P(1)$为真
  - 归纳步骤：证明$\forall k\ (P(k) \rightarrow P(k+1))$
    - //对任意正整数k，给出 $P(k) \vdash P(k+1)$ 的论证步骤.
    - …
  - 因此，对任意正整数n, $P(n)$ 成立. // 即：$\forall n\ P(n)$

数学归纳法（有效性）

- 良序公理
  - 正整数集合的非空子集都有一个最小元素
- 数学归纳法的有效性（归谬法）
  - 假设$\forall n\, P(n)$不成立，则 $\exists n\, (\neg P(n))$成立.
  - 令S=$\{ n \in Z^+ \mid \neg P(n)\}$，S是非空子集.
  - 根据良序公理，S有最小元素，记为m, $m \neq 1$
  - $(m-1) \notin S$, 即$P(m-1)$成立.
  - 根据归纳步骤，$P(m)$成立，即$m \notin S$，矛盾.
  - 因此，$\forall n\, P(n)$成立.

数学归纳法（举例）

- $H_k=1+1/2+\ldots+1/k$ （k为正整数）
- 证明：$H_2^n \geq 1+n/2$ （n为正整数）
  - 基础步骤：$P(1)$为真，$H_2=1+1/2$
  - 归纳步骤：对任意正整数k, $P(k) \Rightarrow P(k+1)$.

$$H_2^{k+1} = H_2^k + 1/(2^k+1)+\ldots+1/2^{k+1}$$

$$\geq (1+k/2)+2^k(1/2^{k+1}) = 1+(1+k)/2$$

  - **因此**，对任意正整数n, $P(n)$ 成立.

数学归纳法（举例）

- 猜测前n个奇数的求和公式，并证明之。
  - 1=1
  - 1+3=4
  - 1+3+5=9
  - 1+3+5+7=16
  - …
  - $1+3+\ldots+(2n-1)=n^2$（n为正整数）

运用数学归纳法时犯的错误

- 平面上任何一组相互间不平行的直线必相交于一点.
  - 基础步骤：$P(2)$为真
  - 归纳步骤：对任意正整数k, $P(k) \vdash P(k+1)$.
    - 前k条交于p_1.
    - 后k条交于p_2.
    - $p_1 = p_2$

强数学归纳法

- 证明目标
  - $\forall n\ P(n)$　//n的论域为正整数集合
- 证明框架
  - 基础步骤：$P(1)$为真
  - 归纳步骤：证明$\forall k\ (P(1)\wedge\ldots\wedge P(k)\rightarrow P(k+1))$
    - //对任意正整数k，给出 $P(1), \ldots, P(k) \vdash P(k+1)$ 的论证步骤.
    - …
  - 因此，对任意正整数n, $P(n)$ 成立. // 即：$\forall n\ P(n)$

强数学归纳法（一般形式）

- 设$P(n)$是与整数n有关的陈述，a和b是两个给定的整数，且$a \le b$.
- 如果能够证明下列陈述
  - $P(a), P(a+1), \ldots, P(b)$.
  - 对任意$k \ge b$, $P(a) \wedge \ldots \wedge P(k) \rightarrow P(k+1)$
- 则下列陈述成立
  - 对任意$n \ge a$, $P(n)$.

强数学归纳法（有效性）

- $\{n \in Z \mid n \geq a\}$是良序的
  - 良序集：该集合的非空子集都有一个最小元素
- 数学归纳法的有效性（归谬法）
  - 假设$\forall n\, P(n)$不成立，则 $\exists n\, (\neg P(n))$成立.
  - 令S=$\{ n \in Z \mid (n \geq a) \wedge \neg P(n) \}$，S是非空子集.
  - 根据良序公理，S有最小元素，记为m, $m>a$
  - $a, \ldots, (m-1) \notin S$, 即$P(a), \ldots, P(m-1)$成立.
  - 根据归纳步骤，$P(m)$成立，即$m \notin S$，矛盾.
  - **因此，**$\forall n\, P(n)$成立.

强数学归纳法（举例）

- 任意整数**n(n ≥2)**可分解为（若干个）素数的乘积
  - **n = 2.**
  - 考察 **n+1.**
- 用**4**分和**5**分就可以组成**12**分及以上的每种邮资.
  - ***P*(12), *P*(13), *P*(14), *P*(15).**
  - 对任意***k*** **≥15,** ***P*(12)∧… ∧*P*(*k*)→*P*(*k*+1)**

Odd Pie Fights

- **Placing an odd number of people in the plane, in such a way that every pair of people has a distinct distance between them. At a signal, each person will throw a pie at the closest other person.**
- **At least one person does not get hit with a pie?**

二项式定理

Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. We have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

We use induction on n. That is, we let

$$P(n) : \forall x \in \mathbb{R},\ (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

We shall prove $P(1)$ is true, and prove

$$\forall k \in \mathbb{N},\ P(k) \implies P(k+1).$$

Again, proof for $P(1)$ is trivial, as $P(1)$ states

$$\forall x \in \mathbb{R},\ 1+x = \binom{1}{0} x^0 + \binom{1}{1} x^1$$

Next, let $k \in \mathbb{N}$ and assume $P(k)$. We prove $P(k+1)$. Let $x \in \mathbb{R}$. We have

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k (1+x) \\ \text{[by I.H.]} &= \left(\sum_{j=0}^{k} \binom{k}{j} x^j \right) (1+x) \\ &= \sum_{j=0}^{k} \binom{k}{j} x^j + \sum_{j=0}^{k} \binom{k}{j} x^{j+1}.\end{aligned}$$

We are going to merge these two sums.

二项式定理

To get an idea, let us consider an explicit example when $k = 2$:

$$\sum_{j=0}^{2} \binom{2}{j} x^j + \sum_{j=0}^{2} \binom{2}{j} x^{j+1}$$
$$= \left[\binom{2}{0} + \binom{2}{1}x + \binom{2}{2}x^2\right] + \left[\binom{2}{0}x + \binom{2}{1}x^2 + \binom{2}{2}x^3\right]$$
$$= \binom{2}{0} + \left[\binom{2}{1} + \binom{2}{0}\right]x + \left[\binom{2}{2} + \binom{2}{1}\right]x^2 + \binom{2}{2}x^3.$$

In general, we have

$$\sum_{j=0}^{k} \binom{k}{j} x^j + \sum_{j=0}^{k} \binom{k}{j} x^{j+1}$$
$$= 1 + \sum_{j=1}^{k} \binom{k}{j} x^j + \sum_{j=1}^{k} \binom{k}{j-1} x^j + x^{k+1}$$
$$= 1 + \sum_{j=1}^{k} \left[\binom{k}{j} + \binom{k}{j-1}\right] x^j + x^{k+1}.$$

Keep in mind that we want to prove this is equal to $\sum_{j=0}^{k+1} \binom{k+1}{j} x^j$, which is

$$1 + \sum_{j=1}^{k} \binom{k+1}{j} x^j + x^{k+1}.$$

Therefore, to finish the proof, it suffices to show that

$$\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}$$

$$\binom{k}{j} + \binom{k}{j-1} = \frac{k!}{j!(k-j)!} + \frac{k!}{(j-1)!(k-j+1)!}$$
$$= \frac{k!}{(j-1)!(k-j)!} \cdot \left[\frac{1}{j} + \frac{1}{k-j+1}\right]$$
$$= \frac{k!}{(j-1)!(k-j)!} \cdot \frac{k+1}{j(k-j+1)}$$
$$= \binom{k+1}{j}.$$

运用良序公理来证明（举例）

- 设a是整数, d是正整数, 则存在唯一的整数q和r满足
  - $0 \le r < d$
  - $a = dq + r$
- 证明
  - 令$S=\{a-dq \mid 0 \le a-dq, q \in Z\}$，S非空.
  - 非负整数集合具有良序性
  - S有最小元，记为$r_0 = a-dq_0$.
  - 可证 $0 \le r_0 < d$
  - 唯一性证明， $0 \le r_1 - r_0 = d(q_0-q_1) < d$，因此，$q_1=q_0$

运用良序公理来证明（举例）

- 在循环赛胜果图中，若存在长度为m（$m\geq 3$）的回路，则必定存在长度为3的回路。

  备注：$a_i \rightarrow a_j$表示a_i赢了a_j

证明

- 设最短回路的长度为k //良序公理的保证
- 假设 $k>3$
- $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \ldots \rightarrow a_k \rightarrow a_1$
- 若$a_3 \rightarrow a_1$, 存在长度为3的回路，矛盾。
- 若$a_1 \rightarrow a_3$, 存在长度为$(k-1)$的回路，矛盾。

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  - 递归定义
  - 结构归纳法与递归算法

递归

递归定义N上的函数

- 递归地定义自然数集合N上的函数
  - 基础步骤：指定这个函数在0处的值；
  - 递归步骤：给出从较小处的值来求出当前的值之规则。

- 举例，阶乘函数F(n)=n!的递归定义
  - F(0)=1
  - F(n)=n·F(n-1) for n>0

Fibonacci 序列

- Fibonacci 序列 $\{f_n\}$ 定义如下
  - $f_0 = 0,$
  - $f_1 = 1,$
  - $f_n = f_{n-1} + f_{n-2}$, 对任意 $n \geq 2.$
- 其前几个数
  - $0, 1, 1, 2, 3, 5, 8, \cdots$
- 证明：对任意 $n \geq 0$，$f_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$

其中，$\alpha = \dfrac{1+\sqrt{5}}{2}, \beta = \dfrac{1-\sqrt{5}}{2}.$

归纳证明: Fibonacci 序列

- 证明：当n=0,1时，陈述正确。

  对于k+1，

$$
\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \\
&= \frac{\alpha^k - \beta^k}{\alpha - \beta} + \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta} \\
&= \frac{\left(\alpha^k + \alpha^{k-1}\right) - \left(\beta^k + \beta^{k-1}\right)}{\alpha - \beta} \\
&= \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}.
\end{aligned}
$$

注意：$\alpha^2 = \alpha + 1$，且$\alpha^{n+1} = \alpha^n + \alpha^{n-1}$ 对任意$n \geq 1$.

类似可证：$n \geq 3$时，$f_n > \alpha^{n-2}$

递归定义集合

- 递归地定义集合
  - 基础步骤：指定一些初始元素；
  - 递归步骤：给出从集合中的元素来构造新元素之规则；
  - 排斥规则（只包含上述步骤生成的那些元素）默认成立

- 举例，正整数集合的子集S
  - $x \in S$
  - 若$x \in S$且$y \in S$，则 $x+y \in S$。

递归定义集合

- 字母表Σ上的字符串集合Σ^*
  - 基础步骤：$\lambda \in \Sigma^*$（λ表示空串）；
  - 递归步骤：若$\omega \in \Sigma^*$且$x \in \Sigma$，则$\omega x \in \Sigma^*$。

- 字符串的长度（Σ^*上的函数l）
  - 基础步骤：$l(\lambda)=0$;
  - 递归步骤：$l(\omega x) = l(\omega) + 1$, 若$\omega \in \Sigma^*$且$x \in \Sigma$

结构归纳法

- 关于递归定义的集合的命题，进行结构归纳证明
  - 基础步骤：证明对于初始元素来说，命题成立；
  - 递归步骤：针对生产新元素的规则，若相关元素满足命题，则新元素也满足命题
- 结构归纳法的有效性源于自然数上的数学归纳法
  - 对运用递归步骤的次数/步数进行归纳
  - 第0步（基础步骤）
  - 第k步推出k+1步（归纳步骤）

例

字符串的长度（Σ^*上的函数l）。

- 基础步骤：$l(\lambda)=0$;
- 递归步骤：$l(\omega x) = l(\omega) +1$, 若$\omega \in \Sigma^*$且$x \in \Sigma$

- 求证：$l(xy) = l(x) + l(y)$, x和y属于Σ^*。
- 证明：
  - 设$P(y)$表示：每当x属于Σ^*，就有$l(xy) = l(x) + l(y)$。
  - 基础步骤：每当x属于Σ^*，就有$l(x\lambda) = l(x) + l(\lambda)$。
  - 递归步骤：假设$P(y)$为真，a属于Σ, 要证$P(ya)$为真。
    - 即证：每当x属于Σ^*，就有$l(xya) = l(x) + l(ya)$
    - $P(y)$为真，$l(xy) = l(x) + l(y)$
    - $l(xya)= l(xy)+1= l(x) + l(y)+1=l(x) + l(ya)$

例（广义归纳法）

- 递归定义$a_{m,n}$
  - $a_{0,0} = 0$
  - $a_{m,n} = a_{m-1,n} + 1 \quad (n=0, m>0)$
  - $a_{m,n} = a_{m,n-1} + n \quad (n>0)$
- 归纳证明 $a_{m,n} = m+n(n+1)/2$

0	**1**	**3**
1	**2**	**4**
2	**3**	**5**

递归算法

An algorithm is called *recursive* if it solves a problem by reducing it to an instance of the same problem with smaller input.

- 举例：欧几里德算法

```
function gcd(a, b)  // a≥b≥0, a>0
  if b=0
        return a
  else
        return gcd(b, a mod b)
```

  - 递归算法的正确性 （数学归纳法）
  - 递归算法的复杂性 （时间、空间）

欧几里德算法的复杂性

- 拉梅定理: 设a和b是满足$a \geq b$的正整数。则欧几里德算法为求出$\gcd(a, b)$而使用除法的次数小于或等于b的十进制位数的5倍。$5(\lfloor \log_{10} b \rfloor + 1)$
- 令$r_0 = a$, $r_1 = b$.
- $r_0 = r_1 q_1 + r_2 \quad 0 \leq r_2 < r_1$
- $r_1 = r_2 q_2 + r_3 \quad 0 \leq r_3 < r_2$
- …
- $r_{n-1} = r_n q_n + r_{n+1} \quad 0 = r_{n+1} < r_n$
- $\gcd(a, b) = r_n$使用了n次除法

- $q_i \geq 1$ for $1 \leq i < n$
- $q_n \geq 2$ because $q_n = r_{n-1}/r_n > 1$
- $r_n \geq 1 = f_2$, $r_{n-1} \geq 2r_n \geq 2 = f_3$
- $b = r_1 \geq r_2 + r_3 \geq f_n + f_{n-1} = f_{n+1} > \alpha^{n-1}$
- $\log_{10} b > (n-1)\log_{10} \alpha$ for $n \geq 2$
- $\log_{10} \alpha > 1/5$

本节小结

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- 递归：
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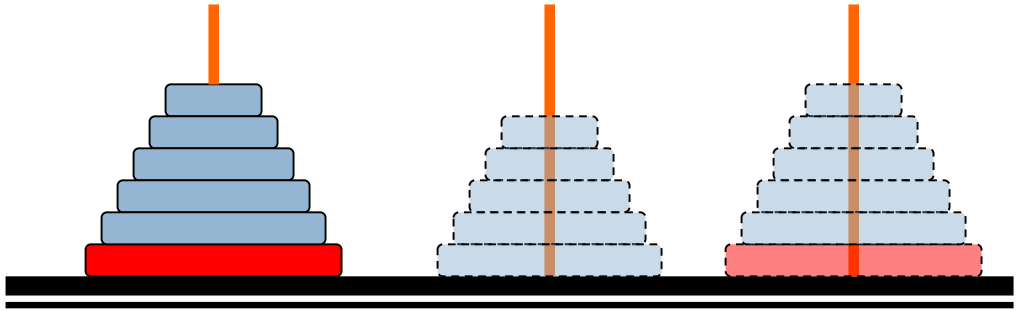
  - 结构归纳法与递归算法

作业

- 见课程网站

Thinking Recursively: Problem 1

- Towers of Hanoi（汉诺塔）
  - How many moves are needed to move all the disks from the first peg to the third peg, under the constraints that we can move only one disk at a time and never place a larger disk on top of a smaller one.



$$T(1) = 1$$
$$T(n) = 2T(n-1) + 1$$

Thinking Recursively: Problem 1

- Towers of Hanoi（汉诺塔）

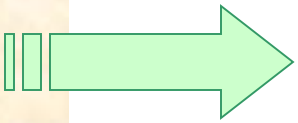
$T(n) = 2T(n-1) + 1$

$2T(n-1) = 4T(n-2) + 2$

$4T(n-2) = 8T(n-3) + 4$

……

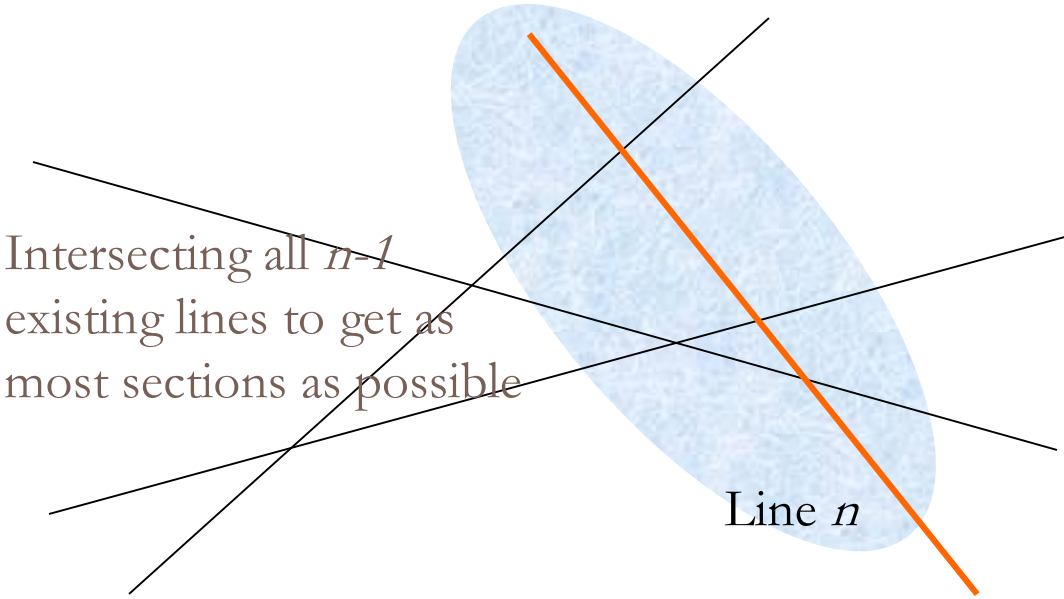
$2^{n-2}T(2) = 2^{n-1}T(1) + 2^{n-2}$



$$T(n) = 2^n - 1$$

Thinking Recursively: Problem 2

- Cutting the plane（平面切割）
  - How many sections can be generated at most by *n* straight lines with infinite length.



$$L(0) = 1$$
$$L(n) = L(n-1) + n$$

Thinking Recursively: Problem 2

- Cutting the plane（平面切割）

$$
\begin{aligned}
L(n) &= L(n-1)+n \\
&= L(n-2)+(n-1)+n \\
&= L(n-3)+(n-2)+(n-1)+n \\
&= \cdots\cdots \\
&= L(0)+1+2+\cdots+(n-2)+(n-1)+n
\end{aligned}
$$

$$L(n) = n(n+1)/2 + 1$$

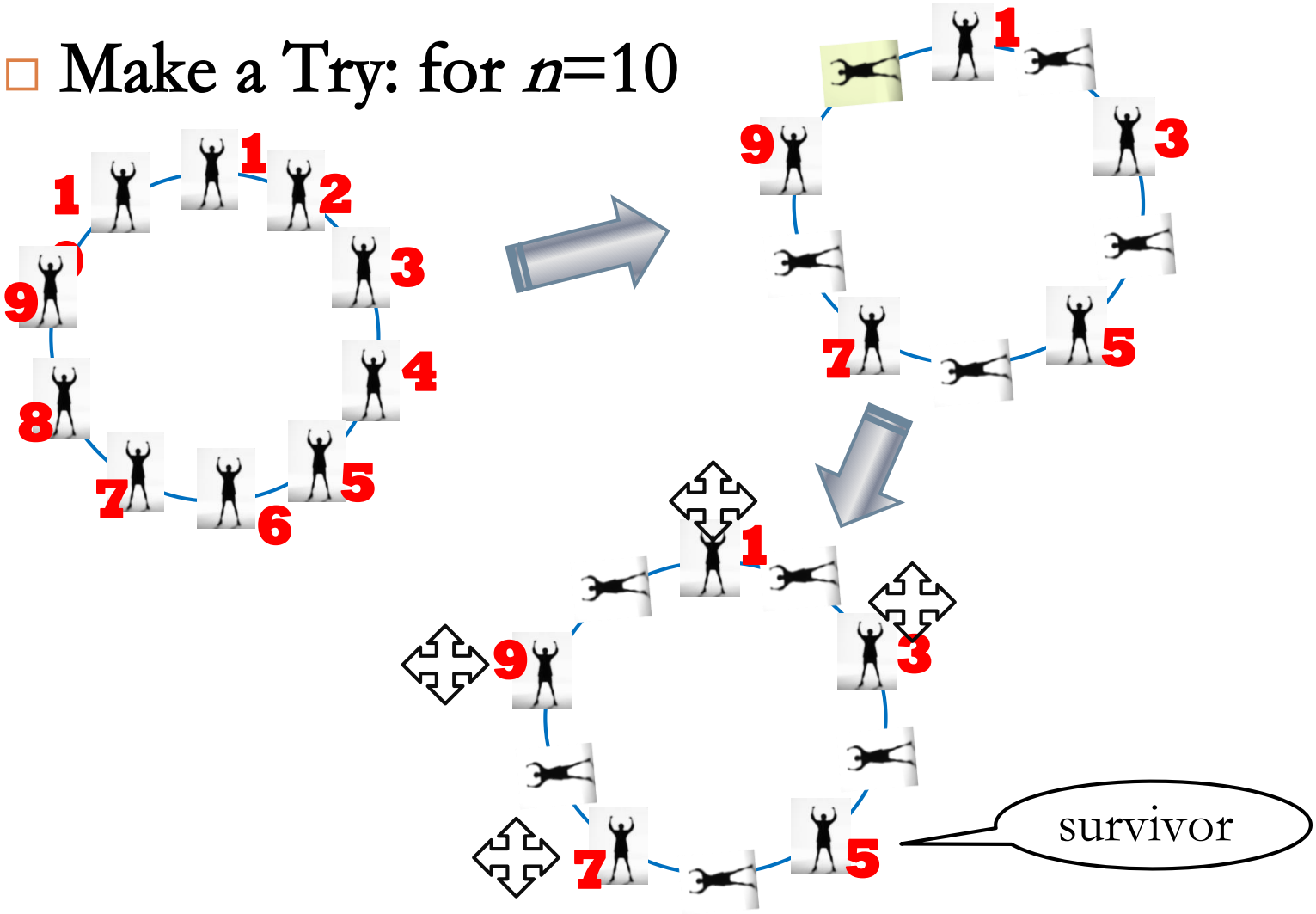
Thinking Recursively: Problem 3

- Josephus Problem: Live or die, it's a problem!
  - Legend has it that Josephus wouldn't have lived to become famous without his mathematical talents. During the Jewish Roman war, he was among a band of 41 Jewish rebels trapped in a cave by the Romans. Preferring suicide to capture, the rebels decided to form a circle and, proceeding around it, to kill every third remaining person until no one was left. But Josephus, along with an unindicted co-conspirator, wanted none of this suicide nonsense; so he quickly calculated where he and his friend should stand in the vicious circle.

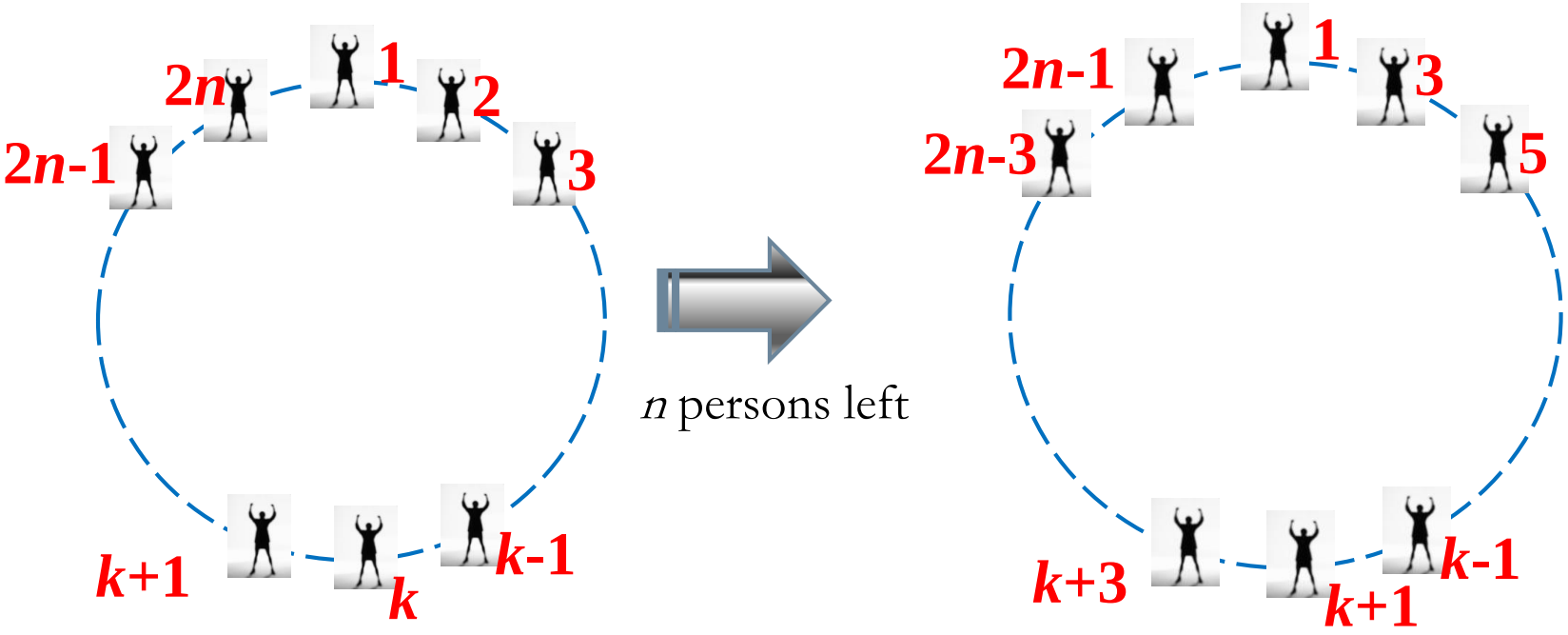
We use a simpler version:
"every second..."

Thinking Recursively: Problem 3

- Make a Try: for n=10



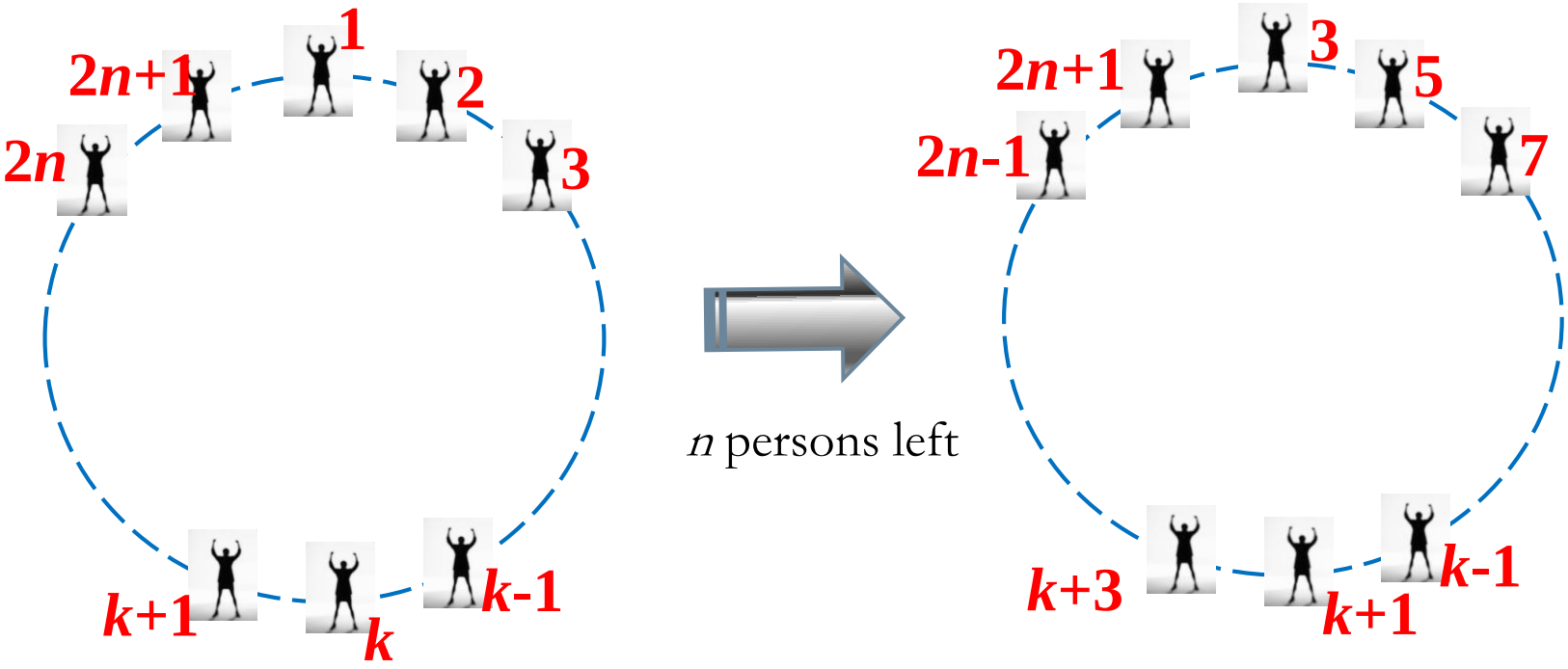
Thinking Recursively: Problem 3



The solution is: newnumber (J(*n*))

And the newnumber(*k*) is **2*k*-1**

Thinking Recursively: Problem 3



The solution is: newnumber (J(*n*))

And for this time, the newnumber(*k*) is $2k+1$

Thinking Recursively: Problem 3

- Solution in Recursive Equations

$$J(1) = 1;$$
$$J(2n) = 2J(n) - 1, \quad \text{for } n \geq 1;$$
$$J(2n+1) = 2J(n) + 1, \quad \text{for } n \geq 1.$$

- Some explicit solutions

Can you find the pattern?

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Thinking Recursively: Problem 3

If we write n in the form $n = 2^m + l$,
(where 2^m is the largest power of 2 not exceeding n and l is what's left),
the solution to our recurrence seems to be:

$$J(2^m + l) = 2l + 1, \quad \text{for } m \geq 0 \text{ and } 0 \leq l < 2^m.$$

As an example: J(100) = J(64+36) = 36*2+1 = 73

Linear Homogeneous Recurrence Relation

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

is called linear homogeneous relation of degree *k*.

$c_n = (-2)c_{n-1}$

$f_n = f_{n-1} + f_{n-2}$

Yes

$a_n = a_{n-1} + 3$

$g_n = g_{n-1}^2 + g_{n-2}$

No

Characteristic Equation

- For a linear homogeneous recurrence relation of degree k

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

the polynomial of degree k

$$x^k = r_1 x^{k-1} + r_2 x^{k-2} + \cdots + r_k$$

is called its characteristic equation.

- The characteristic equation of linear homogeneous recurrence relation of degree 2 is:

$$x^2 - r_1 x - r_2 = 0$$

Solution of Recurrence Relation

- If the characteristic equation $x^2 - r_1 x - r_2 = 0$ of the recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ has two distinct roots s_1 and s_2, then we have the explicit formula for the sequence

$$a_n = u s_1^n + v s_2^n$$

where u and v depend on the initial conditions.

- If the equation has a single root s, then, both s_1 and s_2 in the formula above are replaced by s

Proof of the Solution

Remember the equation : $x^2 - r_1 x - r_2 = 0$

We need prove that : $us_1^n + vs_2^n = r_1 a_{n-1} + r_2 a_{n-2}$

$$
\begin{aligned}
& us_1^n + vs_2^n = us_1^{n-2} s_1^2 + vs_2^{n-2} s_2^2 \\
& = us_1^{n-2}(r_1 s_1 + r_2) + vs_2^{n-2}(r_1 s_2 + r_2) \\
& = r_1 us_1^{n-1} + r_2 us_1^{n-2} + r_1 vs_2^{n-1} + r_2 vs_2^{n-2} \\
& = r_1(us_1^{n-1} + vs_2^{n-1}) + r_2(us_1^{n-2} + vs_2^{n-2}) \\
& = r_1 a_{n-1} + r_2 a_{n-2}
\end{aligned}
$$